

NON-COMMUTATIVE STOCHASTIC DISTRIBUTIONS AND APPLICATIONS TO LINEAR SYSTEMS THEORY

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ABSTRACT. In this paper, we introduce a non-commutative space of stochastic distributions, which contains the non-commutative white noise space, and forms, together with a natural multiplication, a topological algebra. A special inequality which holds in this space allows to characterize its invertible elements and to develop an appropriate framework of non-commutative stochastic linear systems.

1. INTRODUCTION

In this paper we introduce and study a non-commutative version of a space of stochastic distributions, and give applications to mathematical system theory. To set the problem into perspective, recall that, in white noise space analysis, various spaces of stochastic distributions have been introduced by Hida, Kondratiev, and others; see [12] and the references therein. Among those introduced by Kondratiev, one (denoted by \mathcal{S}_{-1}) plays an important role. It is the dual of a Fréchet nuclear space, and in particular the increasing union of a countable family of Hilbert spaces with decreasing norms. \mathcal{S}_{-1} is an algebra when endowed with the Wick product, and the Wick product satisfies in \mathcal{S}_{-1} an inequality, called Våge inequality. The space \mathcal{S}_{-1} was recently used to develop a new approach to the theory of linear stochastic systems, when not only the input is random but also the characteristics of the system. See [1, 4, 3]. We recently defined a large class of topological algebras, which also satisfy a Våge type inequality, and which are furthermore closed under tensor products. See [5]. For the non commutative version of the white noise and of the white noise space we

1991 *Mathematics Subject Classification.* Primary: 16S99, 60H40, 93B07. Secondary: 93A25.

Key words and phrases. convolution algebra, non-commutative white noise space, non-commutative stochastic distributions.

D. Alpay thanks the Earl Katz family for endowing the chair which supported his research, and the Binational Science Foundation Grant number 2010117.

refer to [21]. The non-commutative counterparts of spaces of stochastic distributions, especially ones which satisfy a Våge type inequality, do not seem to have been studied. We begin such a study here, and give applications to non-commutative linear systems parallel to the one done in [1, 4, 3] for the Kondratiev space and in [5] for Våge spaces.

We divide this introduction into three parts. The first two parts are preliminaries about the commutative case, namely on the white noise space and on the Kondratiev space \mathcal{S}_{-1} of stochastic distributions. In the third part we discuss our approach to define a non-commutative space of stochastic distributions and give an outline of the paper.

1.1. The (commutative) white noise space. To set the framework of the commutative case we recall the following definitions. Let \mathcal{H} be a complex Hilbert space. We consider its n -fold Hilbert spaces tensor power $\mathcal{H}^{\otimes n}$. The symmetric product \circ is defined by

$$u_1 \circ \cdots \circ u_n = \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},$$

and the closed subspace of $\mathcal{H}^{\otimes n}$ generated by all vectors of this form is called the n -th symmetric power of \mathcal{H} , and denoted by $\mathcal{H}^{\circ n}$. See [17]. We make the convention $\mathcal{H}^{\otimes 0} = \mathbb{C}$, and the element $1 \in \mathbb{C}$ is called the vacuum vector and denoted by $\mathbf{1}$. Two inner products are defined on $\mathcal{H}^{\circ n}$. The first is called the symmetric inner product, and defined by

$$\langle u_1 \circ \cdots \circ u_n, v_1 \circ \cdots \circ v_n \rangle_{\circ} = \text{per}(\langle u_i, v_j \rangle),$$

where $\text{per}(A)$ is called the permanent of A and has the same definition as a determinant, with the exception that the alternation factor ($\text{sgn}(\sigma)$) is omitted. The second is called the tensor inner product. It is induced by the tensor inner product on $\mathcal{H}^{\otimes n}$

$$\langle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle = \prod_{i=1}^n \langle u_i, v_i \rangle.$$

Therefore, the tensor inner product on $\mathcal{H}^{\circ n}$ can be defined by

$$\langle u_1 \circ \cdots \circ u_n, v_1 \circ \cdots \circ v_n \rangle_{\otimes} = \frac{1}{n!^2} \sum_{\sigma, \tau \in S_n} \langle u_{\sigma(1)}, v_{\tau(1)} \rangle \cdots \langle u_{\sigma(n)}, v_{\tau(n)} \rangle.$$

It is clear that $\|\cdot\|_{\otimes} = \frac{1}{n!} \|\cdot\|_{\circ}$. Assuming $(e_i)_{i \in I}$ is an orthonormal basis of \mathcal{H} , for $\alpha : I \rightarrow \mathbb{N}_0$ with a support $\{i_1, \dots, i_m\}$ ($i_1 < \cdots < i_m$) such that $|\alpha| = \sum_{j=1}^m \alpha_{i_j} = n$, we denote

$$e_{\alpha} = e_{i_1}^{\circ \alpha_{i_1}} \circ \cdots \circ e_{i_m}^{\circ \alpha_{i_m}} \in \mathcal{H}^{\circ n}.$$

(e_α) is clearly an orthogonal basis of $\mathcal{H}^{\circ n}$. The squared symmetric norm of e_α is $\alpha!$, and the squared tensor norm is $\frac{\alpha!}{n!}$.

The symmetric Fock space over \mathcal{H} is the Hilbert space

$$\Gamma^\circ(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circ n},$$

with the corresponding symmetric inner product.

Now let \mathcal{H} be any separable Hilbert space. For the definition of the white noise space, one takes $\mathcal{H} = \mathbf{L}^2(\mathbb{R})$. Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $\mathbf{L}^2(\mathbb{R})$ (for example, the Hermite functions). We define the (commutative) white noise space \mathcal{W} as the symmetric Fock space of $\mathcal{H} = \mathbf{L}^2(\mathbb{R})$. Thus, denoting by ℓ the free commutative monoid generated by \mathbb{N}_0 , that is,

$$\ell = \mathbb{N}_0^{(\mathbb{N})} = \{\alpha \in \mathbb{N}_0^{\mathbb{N}} : \text{supp}(\alpha) \text{ is finite}\},$$

and setting $\nu(\alpha) = \alpha!$ we conclude that

$$\mathcal{W} = \Gamma^\circ(\mathcal{H}) = \left\{ \sum_{\alpha \in \ell} f_\alpha e_\alpha : \sum_{\alpha \in \ell} |f_\alpha|^2 \alpha! < \infty \right\} = L^2(\ell, \nu).$$

For more information on symmetric and non-symmetric Fock spaces we refer to [16, 17].

1.2. The Wick product and the (commutative) Kondratiev space of stochastic distributions. The standard multiplication of two elements in the white noise space is called the Wick product.

Definition 1.1. *The Wick product is defined by $(f, g) \mapsto f \circ g$ whenever it make sense. In terms of the basis, we obtain that*

$$f \circ g = \left(\sum_{\alpha \in \ell} f_\alpha e_\alpha \right) \circ \left(\sum_{\alpha \in \ell} g_\alpha e_\alpha \right) = \sum_{\alpha \in \ell} \left(\sum_{\beta \leq \alpha} f_\beta g_{\alpha-\beta} \right) e_\alpha.$$

As it is obvious from its definition, the Wick product is actually a convolution of functions over the monoid ℓ . It is well known that \mathcal{W} is not closed under it; see Remark 2.6. On the other hand, the dual of the Kondratiev space \mathcal{S}_1 of stochastic test functions, namely the Kondratiev space \mathcal{S}_{-1} of stochastic distributions, is closed under the Wick product. The space \mathcal{S}_1 is defined as follows:

$$\mathcal{S}_1 = \left\{ \sum_{\alpha \in \ell} f_\alpha e_\alpha : \sum_{\alpha \in \ell} |f_\alpha|^2 (2\mathbb{N})^{\alpha p} (\alpha!)^2 < \infty \text{ for all } p \in \mathbb{N} \right\},$$

where $(2\mathbb{N})^\alpha = 2^{\alpha_1} \cdot 4^{\alpha_2} \cdot 6^{\alpha_3} \dots$. It is a countably normed Hilbert space (in the language of Gelfand) which is a subspace of the white noise space \mathcal{W} . Its dual, the Kondratiev space of stochastic distributions \mathcal{S}_{-1} , can be viewed as

$$\begin{aligned} \mathcal{S}_{-1} &= \left\{ \sum_{\alpha \in \ell} f_\alpha e_\alpha : \sum_{\alpha \in \ell} |f_\alpha|^2 (2\mathbb{N})^{-\alpha p} < \infty \text{ for some } p \in \mathbb{N} \right\} \\ &= \bigcup_p \mathbf{L}^2(\ell, \mu_{-p}), \end{aligned}$$

where μ_{-p} is the point measure defined by

$$\mu_{-p}(\alpha) = (2\mathbb{N})^{-\alpha p}.$$

Together with the white noise space these two spaces form the Gelfand triple $(\mathcal{S}_1, \mathcal{W}, \mathcal{S}_{-1})$. These two spaces \mathcal{S}_1 and \mathcal{S}_{-1} are both nuclear (the latter when endowed with the strong topology), a property which allows to consider $\text{Hom}(\mathcal{S}_1, \mathcal{S}_{-1})$ as an appropriate framework for the theory of stochastic linear systems thanks to Schwartz' kernel theorem; see [22, 23] for applications of the latter to the theory of non random linear systems. Furthermore, \mathcal{S}_{-1} is closed under the Wick product. More precisely, the following result holds (see [12]):

Theorem 1.2 (Vågø, 1996). *In the space $\mathcal{S}_{-1} = \bigcup_p \mathbf{L}^2(\ell, \mu_{-p})$ it holds that,*

$$(1.1) \quad \|f \circ g\|_q \leq A_{q-p} \|f\|_p \|g\|_q,$$

(where $\|\cdot\|_p$ denotes the norm of $\mathbf{L}^2(\ell, \mu_{-p})$) for any $q \geq p + 2$, and for any $f \in \mathbf{L}^2(\ell, \mu_{-p})$, $g \in \mathbf{L}^2(\ell, \mu_{-q})$, with

$$A_{q-p} = \left(\sum_{\alpha \in \ell} (2\mathbb{N})^{-\alpha(q-p)} \right)^{\frac{1}{2}} < \infty$$

We note that the finiteness of A_{p-q} was proved by Zhang in [24]. It follows from (1.1) that the multiplication operator

$$M_f : g \mapsto f \circ g$$

is bounded from the Hilbert space $\mathbf{L}^2(\ell, \mu_{-q})$ into itself where $f \in \mathbf{L}^2(\ell, \mu_{-p})$ and $q \geq p + 2$. This also allows us to consider power series. If $\sum_{n=0}^{\infty} a_n z^n$ converges in the open disk with radius R , then for any $f \in \mathbf{L}^2(\ell, \mu_{-p})$ with $\|f\|_p < \frac{R}{A_2}$, we obtain

$$\sum_{n=0}^{\infty} |a_n| \|f^{\circ n}\|_{p+2} \leq \sum_{n=0}^{\infty} |a_n| (A_{q-p} \|f\|_p)^n < \infty,$$

and hence $\sum_{n=0}^{\infty} a_n f^{\circ n} \in \mathbf{L}^2(\ell, \mu_{-(p+2)})$. In this way we are also able to consider the invertible elements of the algebra \mathcal{S}_{-1} . These properties among others, which follows by Våge inequality, are the key tools for the applications described at the beginning.

1.3. The non-commutative case and an outline of the paper.

In a similar way, the non-commutative white noise space is defined by the full Fock space

$$\Gamma(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}^{\otimes n},$$

where again, one takes $\mathcal{H}_0 = \mathbf{L}^2(\mathbb{R})$, but other choices of \mathcal{H}_0 are possible. Denoting by $\widetilde{\ell}$ the free (non-commutative) monoid generated by \mathbb{N} , the space $\widetilde{\mathcal{W}}$ is isometrically isomorphic to $\mathbf{L}^2(\widetilde{\ell}, \nu)$, where ν is now the counting measure (the $\alpha!$ disappeared since we are no longer in the symmetric case). The non-commutative Wick product is defined by $(f, g) \mapsto f \otimes g$, and in view of proposition 2.5, $\widetilde{\mathcal{W}}$ is not closed under it. The counterpart of \mathcal{S}_{-1} is now of the form $\bigcup_p \mathbf{L}^2(\widetilde{\ell}, \widetilde{\mu}_{-p})$ where the measures $\widetilde{\mu}_{-p}$ are defined by (2.1). In the construction of the non-commutative version of the Kondratiev space of stochastic distributions, an inequality similar to the one presented in Theorem 1.2 will be seen to hold.

The outline of the paper is as follows: In Section 2 we construct the non-commutative version of the Kondratiev space, $\widetilde{\mathcal{S}}_{-1}$. In section 3, we discuss about second quantization, and present an inequality which holds in $\widetilde{\mathcal{S}}_{-1}$. Power series, invertible elements and some other properties presented in Section 4. In section 5, we consider $\widetilde{\mathcal{S}}_{-1}$ as an appropriate framework to stochastic linear systems.

2. THE WHITE NOISE SPACE AND THE KONDRATIEV SPACE OF STOCHASTIC DISTRIBUTIONS - THE NON-COMMUTATIVE CASE

To define the non-commutative version of the Gelfand triple $(\mathcal{S}_1, \mathcal{W}, \mathcal{S}_{-1})$, two approaches are possible. In the first one, we replace the free commutative monoid generated by \mathbb{N} , namely ℓ , with the free non-commutative monoid ℓ generated by \mathbb{N} . To ease the notation, we in fact consider a family of (pairwise distinct) symbols $(z_n)_{n \in \mathbb{N}}$ indexed by \mathbb{N} , and consider equivalently the free non-commutative monoid they

generate:

$$\begin{aligned}\tilde{\ell} &= \mathbb{N}^* \\ &\cong \{z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n} : n \in \mathbb{N}, i_1 \neq i_2 \neq \cdots \neq i_n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \mathbb{N}\} \cup \{1\} \\ &\cong \{z_{i_1} z_{i_2} \cdots z_{i_m} : m \in \mathbb{N}, i_1, \dots, i_m \in \mathbb{N}\} \cup \{1\}.\end{aligned}$$

We also consider the induced partial order, that is for $\alpha, \beta \in \tilde{\ell}$, we define $\alpha \leq \beta$ if there exists $\gamma \in \tilde{\ell}$ such that $\alpha\gamma = \beta$.

For $\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n} \in \tilde{\ell}$ (where $i_1 \neq i_2 \neq \cdots \neq i_n$) we define

$$(2\mathbb{N})^\alpha = \prod_{k=1}^n (2i_k)^{\alpha_k} = \prod_{j \in \{i_1, \dots, i_n\}} (2j)^{(\sum_{k: i_k=j} \alpha_k)}.$$

We define the measures $\tilde{\nu}(\alpha) = 1$ for every $\alpha \in \tilde{\ell}$ and for $p \in \mathbb{Z}$,

$$(2.1) \quad \tilde{\mu}_p(\alpha) = (2\mathbb{N})^{\alpha p}.$$

Definition 2.1. We call $\mathbf{L}^2(\tilde{\ell}, \tilde{\nu})$ the non-commutative white noise space and we denote it by $\widetilde{\mathcal{W}}$. Similarly, $\tilde{\mathcal{S}}_1 = \bigcap_{p \in \mathbb{N}} \mathbf{L}^2(\tilde{\ell}, \mu_p)$ and $\tilde{\mathcal{S}}_{-1} = \bigcup_{p \in \mathbb{N}} \mathbf{L}^2(\tilde{\ell}, \mu_{-p})$, topologized as a countably Hilbert space and as its strong dual respectively, will be called the non-commutative Kondratiev space of stochastic test functions and the non-commutative Kondratiev space of stochastic distributions respectively.

In the second approach to consider the non-commutative version of the triple $(\mathcal{S}_1, \mathcal{W}, \mathcal{S}_{-1})$ we replace the symmetric Fock space with the full Fock space. Recall that the full Fock space over \mathcal{H} is the Hilbert space

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}.$$

Assuming $(e_i)_{i \in I}$ is an orthonormal basis of \mathcal{H} , for $\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_m}^{\alpha_m}$ (where $i_1 \neq i_2 \neq \cdots \neq i_m \in I$), such that $|\alpha| = \sum_{j=1}^m \alpha_j = n$, we denote

$$e_\alpha = e_{i_1}^{\otimes \alpha_1} \otimes \cdots \otimes e_{i_m}^{\otimes \alpha_m} \in \mathcal{H}^{\otimes n}.$$

(e_α) is clearly an orthonormal basis of $\mathcal{H}^{\otimes n}$ (with respect to the tensor inner product $\langle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle = \prod_{i=1}^n \langle u_i, v_i \rangle$).

As in the commutative case we make the choice $\mathcal{H} = \mathbf{L}^2(\mathbb{R})$ and denote by $(e_n)_{n \in \mathbb{N}}$ an orthonormal basis of it (e.g. the Hermite functions). For any $p \in \mathbb{Z}$, we denote

$$\mathcal{H}_p = \left\{ \sum_{n=1}^{\infty} f_n e_n : \sum_{n=1}^{\infty} |f_n|^2 (2n)^p < \infty \right\} \cong \mathbf{L}^2(\mathbb{N}, (2n)^p).$$

Remark 2.2. *We note that*

$$\cdots \subseteq \mathcal{H}_2 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_0 \subseteq \mathcal{H}_{-1} \subseteq \mathcal{H}_{-2} \subseteq \cdots,$$

and that $\bigcap_p \mathcal{H}_p$ is the Schwartz space of rapidly decreasing complex smooth functions (in case we indeed choose (e_n) to be the Hermite functions) and $\bigcup_p \mathcal{H}_p$ is its dual, namely the Schwartz space of complex tempered distributions.

Theorem 2.3. *It holds that*

$$\tilde{\mathcal{S}}_1 = \bigcap_{p \in \mathbb{N}} \Gamma(\mathcal{H}_p), \quad \tilde{\mathcal{W}} = \Gamma(\mathcal{H}_0), \quad \text{and} \quad \tilde{\mathcal{S}}_{-1} = \bigcup_{p \in \mathbb{N}} \Gamma(\mathcal{H}_{-p}).$$

Proof. Clearly $((2n)^{-p/2}e_n)$ is an orthonormal basis of \mathcal{H}_p . Hence,

$$\begin{aligned} e_\alpha^{(p)} &= ((2i_1)^{-p/2}e_{i_1})^{\circ \alpha_{i_1}} \circ \cdots \circ ((2i_m)^{-p/2}e_{i_m})^{\circ \alpha_{i_m}} \\ &= \prod_{j=1}^m (2i_j)^{-\alpha_{i_j} p/2} e_\alpha \\ &= (2\mathbb{N})^{-\alpha p/2} e_\alpha \end{aligned}$$

is an orthonormal basis of $\Gamma(\mathcal{H}_p)$. Thus,

$$\Gamma(\mathcal{H}_p) = \left\{ \sum_{\alpha \in \tilde{\ell}} f_\alpha e_\alpha : \sum_{\alpha \in \tilde{\ell}} |f_\alpha|^2 (2\mathbb{N})^{\alpha p} < \infty \right\},$$

and so

$$\begin{aligned} \bigcap_{p \in \mathbb{N}} \Gamma(\mathcal{H}_p) &= \left\{ \sum_{\alpha \in \tilde{\ell}} f_\alpha e_\alpha : \sum_{\alpha \in \tilde{\ell}} |f_\alpha|^2 (2\mathbb{N})^{\alpha p} < \infty \quad \forall p \in \mathbb{N} \right\} \\ &= \bigcap_p \mathbf{L}^2(\tilde{\ell}, \mu_p) \\ &= \tilde{\mathcal{S}}_1, \end{aligned}$$

$$\Gamma(\mathcal{H}_0) = \left\{ \sum_{\alpha \in \tilde{\ell}} f_\alpha e_\alpha : \sum_{\alpha \in \tilde{\ell}} |f_\alpha|^2 < \infty \right\} = \mathbf{L}^2(\tilde{\ell}, \nu) = \tilde{\mathcal{W}},$$

and

$$\begin{aligned} \bigcup_{p \in \mathbb{N}} \Gamma(\mathcal{H}_{-p}) &= \left\{ \sum_{\alpha \in \tilde{\ell}} f_{\alpha} e_{\alpha} : \sum_{\alpha \in \tilde{\ell}} |f_{\alpha}|^2 (2\mathbb{N})^{-\alpha p} < \infty \quad \text{for some } p \in \mathbb{N} \right\} \\ &= \bigcup_p \mathbf{L}^2(\tilde{\ell}, \mu_{-p}) \\ &= \tilde{\mathcal{S}}_{-1}. \end{aligned}$$

□

Definition 2.4. *The Wick product is defined by $(f, g) \mapsto f \otimes g$ whenever it make sense. In terms of the basis we obtain*

$$f \otimes g = \left(\sum_{\alpha \in \ell} f_{\alpha} e_{\alpha} \right) \otimes \left(\sum_{\alpha \in \ell} g_{\alpha} e_{\alpha} \right) = \sum_{\alpha \in \ell} \left(\sum_{\beta \leq \alpha} f_{\beta} g_{\beta^{-1}\alpha} \right) e_{\alpha}.$$

Thus, the Wick product is the convolution of functions over the monoid $\tilde{\ell}$.

Proposition 2.5. *$\tilde{\mathcal{W}}$ is not closed under the Wick product.*

Proof. Let $\iota : \ell^2(\mathbb{N}) \rightarrow \tilde{W}$ be the embedding defined by

$$\langle \iota(f), e_{\alpha} \rangle = \begin{cases} f_n & \text{if } \alpha = z_1^n \\ 0 & \text{otherwise} \end{cases}$$

(where $f = (f_n) \in \ell^2(\mathbb{N})$), and let $f, g \in \ell^2(\mathbb{N})$ such that $\|f * g\| = \infty$, where $*$ denotes the standard convolutions of two elements in $\ell^2(\mathbb{N})$. Then,

$$\|\iota(f) \otimes \iota(g)\| = \|f * g\| = \infty.$$

□

Remark 2.6. *The reason why the commutative white noise space is not closed under the symmetric Wick product is similar. We can simply define $\eta : \ell^2(\mathbb{N}) \rightarrow W$ by*

$$\langle \eta(f), e_{\alpha} \rangle = \begin{cases} f_n / \sqrt{n!} & \text{if } \alpha = (n, 0, 0, \dots) \\ 0 & \text{otherwise} \end{cases}$$

(where $f = (f_n) \in \ell^2(\mathbb{N})$). Thus, for non-negative sequences $f, g \in \ell^2(\mathbb{N})$ such that $\|f * g\| = \infty$,

$$\begin{aligned} \|\eta(f) \otimes \eta(g)\|^2 &= \sum_n \left(\sum_{k=1}^n \frac{1}{\sqrt{k!(n-k)!}} f_k g_{n-k} \right)^2 n! \\ &\geq \sum_n \left(\sum_{k=1}^n f_k g_{n-k} \right)^2 \\ &= \|f * g\|^2 \\ &= \infty. \end{aligned}$$

Similar to the commutative case, it will be shown in the sequel that $\widetilde{\mathcal{S}}_{-1}$ is closed under the Wick product, and moreover it satisfies an inequality similar to the one that was presented in Theorem 1.2.

3. SECOND QUANTIZATION AND AN INEQUALITY OF TENSOR PRODUCT

Let \mathcal{K}_0 be a separable Hilbert space, and let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of \mathcal{K}_0 . Furthermore, let $(a_n)_{n \in \mathbb{N}}$ be a positive sequence of real numbers. For any $p \in \mathbb{Z}$, we denote

$$\mathcal{K}_p = \left\{ \sum_{n=1}^{\infty} f_n e_n : \sum_{n=1}^{\infty} |f_n|^2 a_n^p < \infty \right\} \cong \mathbf{L}^2(\mathbb{N}, a_n^p).$$

We note that

$$\cdots \subseteq \mathcal{K}_2 \subseteq \mathcal{K}_1 \subseteq \mathcal{K}_0 \subseteq \mathcal{K}_{-1} \subseteq \mathcal{K}_{-2} \subseteq \cdots,$$

where the embedding $T_{q,p} : \mathcal{K}_q \hookrightarrow \mathcal{K}_p$ satisfies

$$\|T_{q,p} a_n^{-q/2} e_n\|_p = a_n^{-(q-p)/2} \|a_n^{-p/2} e_n\|_q,$$

and hence

$$\|T_{q,p}\|_{HS} = \sqrt{\sum_{n \in \mathbb{N}} a_n^{-(q-p)}}.$$

The dual of a Fréchet space is nuclear if and only if the initial space is nuclear. Thus, $\bigcup_{p \in \mathbb{N}} \mathcal{K}_{-p}$ is nuclear if and only if $\bigcap_{p \in \mathbb{N}} \mathcal{K}_p$ is nuclear. This in turn will hold if and only if for any p there is some $q > p$ such that $\|T_{q,p}\|_{HS} < \infty$, that is, if and only if there exists some $d > 0$ such that $\sum_{n \in \mathbb{N}} a_n^{-d}$ converges. We note that in this case, d can be chosen so that

$$\sum_{n \in \mathbb{N}} a_n^{-d} < 1.$$

We call the smallest integer d which satisfy this inequality the index of $\bigcup_{p \in \mathbb{N}} \mathcal{K}_{-p}$. In this section we show that if $\bigcup_{p \in \mathbb{N}} \mathcal{K}_{-p}$ is nuclear of index d , then $\bigcup_{p \in \mathbb{N}} \Gamma(\mathcal{K}_{-p})$ has the property that

$$\|f \otimes g\|_q \leq \|\Gamma(T_{q,p})\|_{HS} \|f\|_p \|g\|_q \quad \text{for all } q \geq p + d,$$

where $\|\cdot\|_p$ is the norm associated to $\Gamma(\mathcal{K}_{-p})$, and $\|\Gamma(T_{q,p})\|_{HS}$ is finite. The case $a_n = 2n$ (and hence $d = 2$) corresponds to the non-commutative Kondratiev space, and is discussed in the next section.

Let $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator between two Hilbert spaces. Then $T^{\otimes n} : \mathcal{H}_1^{\otimes n} \rightarrow \mathcal{H}_2^{\otimes n}$, defined by

$$T^{\otimes n}(u_1 \otimes \cdots \otimes u_n) = Tu_1 \otimes \cdots \otimes Tu_n,$$

is a bounded linear operator between $\mathcal{H}_1^{\otimes n}$ and $\mathcal{H}_2^{\otimes n}$. When T is a contraction, it induces a bounded linear operator $\Gamma(\mathcal{H}_1) \rightarrow \Gamma(\mathcal{H}_2)$, denoted by $\Gamma(T)$, and called the second quantization of T .

Let (λ_n) be a sequence of non-negative numbers. For $\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n} \in \tilde{\ell}$ (where $i_1 \neq i_2 \neq \cdots \neq i_n$) we denote

$$\lambda_{\mathbb{N}}^{\alpha} = \prod_{k=1}^n \lambda_{i_k}^{\alpha_k} = \prod_{j \in \{i_1, \dots, i_n\}} \lambda_j^{(\sum_{k: i_k=j} \alpha_k)}.$$

We recall that if $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a compact operator, then

$$Tf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle h_n$$

where $(e_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ are orthonormal basis of \mathcal{H}_1 and \mathcal{H}_2 respectively and where (λ_n) is a non-negative sequence converges to zero. Conversely, any such a decomposition defines a compact operator $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ (see for instance [18]).

Theorem 3.1. *Let $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a compact operator with*

$$Tf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle h_n$$

where $(e_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ are orthonormal basis of \mathcal{H}_1 and \mathcal{H}_2 respectively and where (λ_n) is a non-negative sequence converging to zero. Then,

(a) It holds that

$$\Gamma(T)f = \sum_{\alpha \in \tilde{\ell}} \lambda_{\mathbb{N}}^{\alpha} \langle f, e_{\alpha} \rangle h_{\alpha},$$

where $(e_\alpha)_{\tilde{\ell}}$ and $(h_\alpha)_{\tilde{\ell}}$ are orthonormal basis of $\Gamma(\mathcal{H}_1)$ and $\Gamma(\mathcal{H}_2)$ respectively.

(b) If furthermore T is an Hilbert-Schmidt operator, i.e. $(\lambda_n) \in \ell^2(\mathbb{N})$, then

$$\|\Gamma(T)\|_{HS}^2 = \sum_{n=0}^{\infty} \|T\|_{HS}^{2n}.$$

In particular, $\Gamma(T)$ is a Hilbert-Schmidt operator if and only if T is a Hilbert-Schmidt operator with $\|T\|_{HS} < 1$ and in this case we obtain

$$\|\Gamma(T)\|_{HS} = \frac{1}{\sqrt{1 - \|T\|_{HS}^2}}$$

Proof. For any $\alpha \in \tilde{\ell}$ let $e_\alpha = e_{i_1}^{\otimes \alpha_1} \otimes \dots \otimes e_{i_m}^{\otimes \alpha_m}$ and $h_\alpha = h_{i_1}^{\otimes \alpha_1} \otimes \dots \otimes h_{i_m}^{\otimes \alpha_m}$. Then, $(e_\alpha)_{\alpha \in \tilde{\ell}}$ and $(h_\alpha)_{\alpha \in \tilde{\ell}}$ are orthonormal basis of \mathcal{H}_1 and \mathcal{H}_2 respectively.

(a) We have that

$$\begin{aligned} \Gamma(T)e_\alpha &= (Te_{i_1})^{\otimes \alpha_1} \otimes \dots \otimes (Te_{i_m})^{\otimes \alpha_m} \\ &= (\lambda_{i_1} h_{i_1})^{\otimes \alpha_1} \otimes \dots \otimes (\lambda_{i_m} h_{i_m})^{\otimes \alpha_m} \\ &= \lambda_{\mathbb{N}}^\alpha h_\alpha. \end{aligned}$$

Thus, by linearity

$$\Gamma(T)f = \sum_{\alpha \in \tilde{\ell}} \lambda_{\mathbb{N}}^\alpha \langle f, e_\alpha \rangle h_\alpha.$$

(b) We have that

$$\begin{aligned} \|\Gamma(T)\|_{HS}^2 &= \sum_{\alpha \in \tilde{\ell}} \|\Gamma(T)e_\alpha\|^2 \\ &= \sum_{n=0}^{\infty} \sum_{\alpha \in \tilde{\ell}, |\alpha|=n} \|T^{\otimes n} e_\alpha\|^2 \\ &= \sum_{n=0}^{\infty} \sum_{\alpha \in \tilde{\ell}, |\alpha|=n} \prod_{i=1}^{\infty} \|Te_i\|^{2\alpha_i} \\ &= \sum_{n=0}^{\infty} \sum_{\alpha \in \tilde{\ell}, |\alpha|=n} \frac{n!}{\alpha!} \prod_{i=1}^{\infty} \|Te_i\|^{2\alpha_i}. \end{aligned}$$

Considering an experiment with \mathbb{N} results, where the probability of the result i is $p_i = \|T\|_{HS}^{-2} \|Te_i\|^2$ (and so $\sum p_i = 1$), the probability

that repeating the experiment n times yields that the result i occurs α_i times for any i is

$$\frac{n!}{\alpha!} \prod_{i=1}^{\infty} p_i^{\alpha_i} = \|T\|_{HS}^{-2n} \frac{n!}{\alpha!} \prod_{i=1}^{\infty} \|Te_i\|^{2\alpha_i}.$$

Thus,

$$\sum_{\alpha \in \ell, |\alpha|=n} \frac{n!}{\alpha!} \prod_{i=1}^{\infty} \|Te_i\|^{2\alpha_i} = \|T\|_{HS}^{2n},$$

and we obtain the requested result. \square

Theorem 3.2. *If $\bigcup_{p \in \mathbb{N}} \mathcal{K}_{-p}$ is nuclear of index d , then $\bigcup_{p \in \mathbb{N}} \Gamma(\mathcal{K}_{-p})$ is nuclear and has the property that*

$$\|f \otimes g\|_q \leq \|\Gamma(T_{q,p})\|_{HS} \|f\|_p \|g\|_q \quad \text{for all } q \geq p + d,$$

where $\|\cdot\|_p$ is the norm associated to $\Gamma(\mathcal{K}_{-p})$, and where

$$\|T_{q,p}\|_{HS} = \sum_{\alpha \in \tilde{\ell}} a_{\mathbb{N}}^{-\alpha(q-p)} = \frac{1}{\sqrt{1 - \sum_{n \in \mathbb{N}} a_n^{-(q-p)}}}.$$

Proof. Denoting $b_{\alpha} = a_{\mathbb{N}}^{\alpha}$, we have that

$$\Gamma(\mathcal{K}_{-p}) = \left\{ (f_{\alpha})_{\alpha \in \tilde{\ell}} : \sum_{\alpha \in \tilde{\ell}} |f_{\alpha}|^2 b_{\alpha}^{-p} < \infty \right\}.$$

Since $\bigcup_{p \in \mathbb{N}} \mathcal{K}_{-p}$ is nuclear of index d ,

$$\|T_{q,p}\|^2 = \sum_{n \in \mathbb{N}} a_n^{-(q-p)} < 1 \quad \text{for any } q \geq p + d$$

In view of Theorem 3.1, $\Gamma(T_{q,p})$ is Hilbert-Schmidt and

$$\sum_{\alpha \in \tilde{\ell}} b_{\alpha}^{-(q-p)} = \sum_{\alpha \in \tilde{\ell}} a_{\mathbb{N}}^{-\alpha(q-p)} = \|\Gamma(T_{q,p})\|_{HS}^2 = \frac{1}{1 - \|T_{q,p}\|_{HS}^2} < \infty.$$

Since for any $\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n} \in \tilde{\ell}$ and $\beta = z_{j_1}^{\beta_1} z_{j_2}^{\beta_2} \cdots z_{j_m}^{\beta_m} \in \tilde{\ell}$ it holds that

$$b_{\alpha} b_{\beta} = a_{\mathbb{N}}^{\alpha} a_{\mathbb{N}}^{\beta} = \prod_{k=1}^n a_{i_k}^{\alpha_k} \cdot \prod_{l=1}^m a_{j_l}^{\beta_l} = a_{\mathbb{N}}^{\alpha\beta} = b_{\alpha\beta},$$

for any $f \in \Gamma(\mathcal{H}_{-p})$ and $g \in \Gamma(\mathcal{H}_{-q})$ we obtain

$$\begin{aligned}
\|f \otimes g\|_q^2 &= \sum_{\gamma \in \tilde{\ell}} \left| \sum_{\alpha \leq \gamma} f_\alpha g_{\alpha^{-1}\gamma} b_\gamma^{-q/2} \right|^2 \\
&= \sum_{\gamma \in \tilde{\ell}} \left(\sum_{\alpha \leq \gamma} |f_\alpha| b_\alpha^{-q/2} |g_{\alpha^{-1}\gamma}| b_{\alpha^{-1}\gamma}^{-q/2} \right)^2 \\
&= \sum_{\gamma \in \tilde{\ell}} \left(\sum_{\alpha, \alpha' \leq \gamma} |f_\alpha| b_\alpha^{-q/2} |f_{\alpha'}| b_{\alpha'}^{-q/2} |g_{\alpha^{-1}\gamma}| b_{\alpha^{-1}\gamma}^{-p/2} |g_{\alpha^{-1}\gamma'}| b_{\alpha^{-1}\gamma'}^{-q/2} \right) \\
&\leq \sum_{\alpha, \alpha' \in \tilde{\ell}} \left(|f_\alpha| b_\alpha^{-q/2} |f_{\alpha'}| b_{\alpha'}^{-q/2} \sum_{\gamma \geq \alpha, \alpha'} |g_{\alpha^{-1}\gamma}| b_{\alpha^{-1}\gamma}^{-q/2} |g_{\alpha^{-1}\gamma'}| b_{\alpha^{-1}\gamma'}^{-q/2} \right) \\
&\leq \left(\sum_{\beta \in \tilde{\ell}} |f_\beta| b_\beta^{-p/2} \right)^2 \left(\sum_{\beta \in \tilde{\ell}} |g_\beta|^2 b_\beta^{-q} \right)^{\frac{1}{2}} \left(\sum_{\beta \in \tilde{\ell}} |g_\beta|^2 b_\beta^{-q} \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\beta \in \tilde{\ell}} b_\beta^{-(q-p)} \right) \left(\sum_{\beta \in \tilde{\ell}} |f_\beta|^2 b_\beta^{-p} \right) \left(\sum_{\beta \in \tilde{\ell}} |g_\beta|^2 b_\beta^{-q} \right) \\
&= \|\Gamma(T_{q,p})\|_{HS}^2 \|f\|_p^2 \|g\|_q^2.
\end{aligned}$$

Thus, we proved the requested result. \square

4. THE ALGEBRA OF THE NON-COMMUTATIVE KONDRATIEV SPACE OF STOCHASTIC DISTRIBUTIONS

We now specialize the results of the preceding section to $a_n = 2n$, and denote by \mathcal{H}_p the corresponding spaces:

$$\mathcal{H}_p = \left\{ \sum_{n=1}^{\infty} f_n e_n : \sum_{n=1}^{\infty} |f_n|^2 (2n)^p < \infty \right\} \cong \mathbf{L}^2(\mathbb{N}, (2n)^p),$$

Denoting by $T_{q,p}$ the embedding $\mathcal{H}_q \hookrightarrow \mathcal{H}_p$, it holds that

$$\|T_{q,p}\|_{HS}^2 = \sum_{n \in \mathbb{N}} (2n)^{-(q-p)} = 2^{-(q-p)} \zeta(q-p),$$

where ζ denotes Riemann's zeta function. Since for any $s \geq 2$, $\zeta(s) < 2^s$, for any $q \geq p+2$, $\|T_{q,p}\|_{HS} < 1$. In view of Theorems 3.1 and 3.2 we obtain the following result:

Theorem 4.1.

- (a) The non-commutative Kondratiev spaces $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_{-1}$ are both nuclear spaces.
- (b) For any $q \geq p + 2$

$$B_{p-q}^2 = \sum_{\alpha \in \tilde{\ell}} (2\mathbb{N})^{-\alpha(q-p)} = \frac{1}{1 - 2^{-(q-p)} \zeta(q-p)}.$$

- (c) For any $q \geq p + 2$ and for any $f \in \Gamma(\mathcal{H}_{-p})$ and $g \in \Gamma(\mathcal{H}_{-q})$

$$(4.1) \quad \|f \otimes g\|_q \leq B_{q-p} \|f\|_p \|g\|_q$$

where $\|\cdot\|_p$ is the norm associated to $\Gamma(\mathcal{H}_{-p})$.

We now show that the non-commutative Wick product is continuous. We first need the following proposition.

Proposition 4.2. *Let $(f_\lambda)_{\lambda \in \Lambda}$ be a net in $\tilde{\mathcal{S}}_{-1}$. Then $f_\lambda \rightarrow f$ in the strong topology if and only if there exists $p \in \mathbb{N}$ such that $f_\lambda, f \in \Gamma(\mathcal{H}_{-p})$ and $f_\lambda \rightarrow f$ in the strong topology of $\Gamma(\mathcal{H}_{-p})$.*

Proof. Suppose $f_\lambda \rightarrow f$ in the strong topology of $\tilde{\mathcal{S}}_{-1}$. In particular, $\{f_\lambda\}_{\lambda \in \Lambda} \cup \{f\}$ is strongly bounded. Therefore, there exists $p \in \mathbb{N}$ such that

$$\{f_\lambda\}_{\lambda \in \Lambda} \cup \{f\} \subseteq \Gamma(\mathcal{H}_{-p}) = \Gamma(\mathcal{H}_p)'$$

(see [9, §5.3 p. 45]). Let B be a bounded set in $\Gamma(\mathcal{H}_p)$; then $B \cap \mathcal{S}_1$ is a dense subset of B . Therefore,

$$\sup_{\varphi \in B} |f_\lambda(\varphi) - f(\varphi)| = \sup_{\varphi \in B \cap \Gamma(\mathcal{H}_p)} |f_\lambda(\varphi) - f(\varphi)| \rightarrow 0.$$

Thus, $f_\lambda \rightarrow f$ in the strong topology of $\Gamma(\mathcal{H}_{-p})$. The opposite direction is clear. \square

Theorem 4.3. *The Wick product is a continuous function $\tilde{\mathcal{S}}_{-1} \times \tilde{\mathcal{S}}_{-1} \rightarrow \tilde{\mathcal{S}}_{-1}$ in the strong topology. Hence $(\tilde{\mathcal{S}}_{-1}, +, \otimes)$ is a topological \mathbb{C} -algebra.*

Proof. Assuming $((f_\lambda, g_\lambda))_{\lambda \in \Lambda}$ is a net which converges to (f, g) in the strong topology of $\tilde{\mathcal{S}}_{-1} \times \tilde{\mathcal{S}}_{-1}$, then in particular, $f_\lambda \rightarrow f$ and $g_\lambda \rightarrow g$ in the strong topology of $\tilde{\mathcal{S}}_{-1}$. According to Proposition 4.2, there exist $p, q \in \mathbb{N}$ such that $f_\lambda, f \in \Gamma(\mathcal{H}_{-p})$ and $g_\lambda, g \in \Gamma(\mathcal{H}_{-q})$ where $f_\lambda \rightarrow f$ in the strong topology of $\Gamma(\mathcal{H}_{-p})$ and $g_\lambda \rightarrow g$ in the strong topology of $\Gamma(\mathcal{H}_{-q})$. We may assume that $p \geq q + 2$. In view of Theorem 3.1, $f \otimes g_\lambda, f \otimes g \in \Gamma(\mathcal{H}_{-q})$, and $\otimes : \Gamma(\mathcal{H}_{-p}) \times \Gamma(\mathcal{H}_{-q}) \rightarrow \Gamma(\mathcal{H}_{-q})$ is continuous. Since $(f_\lambda, g_\lambda) \rightarrow (f, g)$ in the strong topology of $\Gamma(\mathcal{H}_{-p}) \times \Gamma(\mathcal{H}_{-q})$, $f_\lambda \otimes g_\lambda \rightarrow f \otimes g$ in the strong topology of $\Gamma(\mathcal{H}_{-q})$. Again, using Proposition 4.2, we have that $f_\lambda \otimes g_\lambda \rightarrow f \otimes g$ in the strong topology of $\tilde{\mathcal{S}}_{-1}$. Thus the Wick product is strongly continuous. \square

Definition 4.4. Let $f = \sum_{\alpha \in \tilde{\ell}} f_{\alpha} e_{\alpha} \in \tilde{\mathcal{S}}_{-1}$. Then, $f_0 \in \mathbb{C}$ is called the generalized expectation of f and is denoted by $E[f]$.

From this definition we have

$$E[f \otimes g] = E[f]E[g] \quad \text{and} \quad E[1_{\tilde{\mathcal{S}}_{-1}}] = 1_{\mathbb{C}} \quad \forall f, g \in \mathcal{S}_{-1}.$$

Thus, $E : \tilde{\mathcal{S}}_{-1} \rightarrow \mathbb{C}$ is a unital algebra homomorphism. In the sequel, we will see it is the only homomorphism with this property (see Proposition 4.9).

Proposition 4.5. Let M be a positive number. Then, for any $f \in \tilde{\mathcal{S}}_{-1}$ such that $E[f] = 0$, it holds that $\lim_{q \rightarrow \infty} \|f\|_q = 0$.

Proof. Let $f = \sum_{\alpha \in \tilde{\ell}} f_{\alpha} e_{\alpha} \in \Gamma(\mathcal{H}_{-p})$ with $f_0 = 0$. Then for all $\alpha \in \tilde{\ell}$ we have

$$\lim_{q \rightarrow \infty} |f_{\alpha}|^2 (2\mathbb{N})^{-q\alpha} = 0,$$

and for all $q > p$,

$$|f_{\alpha}|^2 (2\mathbb{N})^{-q\alpha} \leq |f_{\alpha}|^2 (2\mathbb{N})^{-p\alpha},$$

where $\sum_{\alpha \in \tilde{\ell}} |f_{\alpha}|^2 a_{\alpha}^{-p} = \|f\|_p^2 < \infty$. Thus, the dominated convergence theorem implies

$$\lim_{q \rightarrow \infty} \|f\|_q^2 = \lim_{q \rightarrow \infty} \sum_{\alpha \in \tilde{\ell}} |f_{\alpha}|^2 (2\mathbb{N})^{-q\alpha} = \sum_{\alpha \in \tilde{\ell}} \lim_{q \rightarrow \infty} |f_{\alpha}|^2 (2\mathbb{N})^{-q\alpha} = 0.$$

□

Proposition 4.6. Let f be in $\Gamma(\mathcal{H}_{-p})$. Then

$$f^{\otimes n} \in \Gamma(\mathcal{H}_{-(p+2)}) \quad \forall n \in \mathbb{N}.$$

Moreover,

$$\|f^{\otimes n}\|_{p+2} \leq B_2^n \|f\|_p^n.$$

Proof. Obviously, $f^0 = 1 \in \Gamma(\mathcal{H}_{-(p+2)})$, and $\|f^0\|_{p+2} = A(2)^0 \|f\|_p^0$. By induction,

$$\begin{aligned} \|f^{\otimes(n+1)}\|_{p+2} &= \|f \otimes f^{\otimes n}\|_{p+2} \\ &\leq B_2 \|f\|_p \|f^{\otimes n}\|_{p+2} \\ &\leq B_2^n \|f\|_p^{n+1} < \infty \end{aligned}$$

□

More generally, given a polynomial $p(z) = \sum_{n=0}^N p_n z^n$ ($p_n \in \mathbb{C}$), we define its Wick version $p : \tilde{\mathcal{S}}_{-1} \rightarrow \tilde{\mathcal{S}}_{-1}$ by

$$p(f) = \sum_{n=0}^N p_n f^{\otimes n}$$

By Proposition 4.6, we have that $p(f) \in \tilde{\mathcal{S}}_{-1}$ for $f \in \tilde{\mathcal{S}}_{-1}$. The following proposition considers the case of power series.

Proposition 4.7. *Let $\phi(z) = \sum_{n \in \mathbb{N}} \phi_n z^n$ be a power series (with complex coefficients) which converges absolutely in the open disk with radius R . Then for any $f \in \tilde{\mathcal{S}}_{-1}$ such that $|E[f]| < \frac{R}{B_2}$ it holds that*

$$\phi(f) = \sum_{n \in \mathbb{N}} \phi_n f^{\otimes n} \in \tilde{\mathcal{S}}_{-1}.$$

Proof. Applying Proposition 4.5, there exists q such that

$$\|f - E(f)\|_q < \frac{R}{B_2} - |E[f]|.$$

Therefore,

$$\|f\|_q \leq \|f - E(f)\|_q + |E(f)| < \frac{R}{B_2}.$$

By Proposition 4.6, for all $p \geq q + 2$,

$$\begin{aligned} \sum_{n \in \mathbb{N}} |\phi_n| \|f^{\otimes n}\|_p &\leq \sum_{n \in \mathbb{N}} |\phi_n| B_2^n \|f\|_q^n \\ &= \sum_{n \in \mathbb{N}} |\phi_n| (B_2 \|f\|_q)^n \\ &< \infty. \end{aligned}$$

Since $\Gamma(\mathcal{H}_{-p})$ is a Hilbert space, $\phi(f) = \sum_{n \in \mathbb{N}} \phi_n f^{\otimes n} \in \Gamma(\mathcal{H}_{-p})$. Thus, $\phi(f) \in \tilde{\mathcal{S}}_{-1}$. \square

Proposition 4.8. *An element $f \in \Gamma(\mathcal{H}_{-p})$ is invertible if and only if $E[f]$ is invertible.*

Proof. If $E[f] \neq 0$, we can assume that $E[f] = 1$. By Proposition 4.7 we have that $\sum_{n \in \mathbb{N}} (1 - f)^{\otimes n} \in \tilde{\mathcal{S}}_{-1}$. Furthermore,

$$f \otimes \left(\sum_{n \in \mathbb{N}} (1 - f)^{\otimes n} \right) = 1.$$

Conversely, assume f invertible. Then there exists $f^{-1} \in \tilde{\mathcal{S}}_{-1}$ such that $f \otimes f^{-1} = 1$. Hence, $E[f]E[f^{-1}] = E[f \otimes f^{-1}] = 1$. \square

Proposition 4.9. *The following properties hold:*

- (a) $GL(\tilde{\mathcal{S}}_{-1})$ is open.
- (b) The spectrum of $f \in \tilde{\mathcal{S}}_{-1}$, $\sigma(f) = \{\lambda \in \mathbb{C} : f - \lambda \text{ is not invertible}\}$ is the singleton $\{E[f]\}$.
- (c) E is unique as a homomorphism $\mathcal{R} \rightarrow \mathbb{C}$ mapping $1_{\tilde{\mathcal{S}}_{-1}}$ to $1_{\mathbb{C}}$.

Proof.

- (a) By Proposition 4.8, we have that $\{f \in \tilde{\mathcal{S}}_{-1} : E[f] \neq 0\}$ is the set of all invertible elements in \mathcal{R} . In other words, $GL(\tilde{\mathcal{S}}_{-1}) = E^{-1}(GL(\mathbb{C}))$. In particular, since E is continuous, $GL(\tilde{\mathcal{S}}_{-1})$ is open.
- (b) Clearly, $f - \lambda 1$ does not have an inverse if and only if $\lambda = E(f)$.
- (c) Let $\varphi : \tilde{\mathcal{S}}_{-1} \rightarrow \mathbb{C}$ be a homomorphism mapping $1_{\tilde{\mathcal{S}}_{-1}}$ to $1_{\mathbb{C}}$ and let $f \in \tilde{\mathcal{S}}_{-1}$. Since $\varphi(f - \varphi(f)) = 0$, $\varphi(f) \in \sigma(f)$, that is $\varphi(f) = E[f]$. \square

5. APPLICATIONS TO NON-COMMUTATIVE LINEAR SYSTEMS

We refer to [7, 11, 14, 19] for general information on the theory of linear systems, including over commutative rings, and to the papers [10, 20] for more information on linear system on non-commutative rings, and in particular for the notions of controllable and observable pairs. In the present setting an input-output system will be a map of the form now an input-output relation of the form

$$(5.1) \quad y_n = \sum_{m=0}^n h_m \otimes u_{n-m}, \quad n \in \mathbb{N}_0,$$

where the input sequence $(u_n)_{n \in \mathbb{N}_0}$, the impulse response $(h_n)_{n \in \mathbb{N}_0}$ belong to $\tilde{\mathcal{S}}_{-1}^{q \times 1}$ and $\tilde{\mathcal{S}}_{-1}^{p \times q}$ respectively. Then, the output sequence belongs to $\tilde{\mathcal{S}}_{-1}^{p \times 1}$. When the impulse response (h_n) or the input sequence (u_n) are not random, the Wick product reduces to the pointwise product of complex numbers, and we recover classical convolution systems. The transfer function of the system (5.1) is (the possibly divergent) series defined by

$$\mathcal{H}(z) = \sum_{n=0}^{\infty} h_n z^n,$$

where z is a complex variable. The realization problem in this setting is to find, when possible, realization of \mathcal{H} in the form

$$(5.2) \quad \mathcal{H}(z) = D + zC \otimes (I - zA)^{-1}B,$$

where A, B, C and D are matrices of appropriate entries and with entries in $\tilde{\mathcal{S}}_{-1}$, and

$$(I - zA)^{-1} = \sum_{k=0}^{\infty} z^k A^{\otimes k}.$$

The series converges in a neighborhood of the origin thanks to Proposition 4.7.

The results presented in [1, 4, 3] for the case of the commutative Kondratiev space \mathcal{S}_{-1} of stochastic distributions still hold for the non-commutative case because of the underlying structure and in particular of inequality (4.1). We will present here one representative result, see Theorem 5.2. Note that the arguments in [1, 4, 3] are in the setting of power series (because one considers there the Hermite transform of the Kondratiev space rather than the Kondratiev space itself), and make use of derivatives. For the general case, when no power series are available, we need to introduce and prove the continuity, of the operators D_m , $m = 1, 2, \dots$ defined by

$$D_m(z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n}) = \sum_{\{j: i_j = m, \alpha_j > 0\}} \alpha_j z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_{(j-1)}}^{\alpha_{(j-1)}} z_{i_j}^{\alpha_j - 1} z_{i_{(j+1)}}^{\alpha_{(j+1)}} \cdots z_{i_n}^{\alpha_n},$$

where, to ease the notation, we write $z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n}$ instead of $e_{z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_n}^{\alpha_n}}$, and extend by linearity to any finite linear combination of such elements, and prove that these operators are continuous.

Proposition 5.1. *D_m is a well defined continuous linear operator $\tilde{\mathcal{S}}_{-1} \rightarrow \tilde{\mathcal{S}}_{-1}$ and it holds that*

$$D_m(f \otimes g) = D_m(f) \otimes g + f \otimes D_m(g)$$

for any $f, g \in \tilde{\mathcal{S}}_{-1}$.

Proof. Let $f = \sum_{\alpha \in \tilde{\ell}} f_{\alpha} e_{\alpha} \in \tilde{\mathcal{S}}_{-1}$. Then there exists $p \in \mathbb{N}$ such that

$$\sum_{\alpha \in \tilde{\ell}} |f_{\alpha}|^2 (2\mathbb{N})^{-\alpha p} < \infty.$$

For any $0 \leq j \leq n$, let r_j be defined by

$$r_j : \{\alpha \in \tilde{\ell} : |\alpha| = n\} \rightarrow \{\alpha \in \tilde{\ell} : |\alpha| = n+1\}$$

defined by

$$r_j(z_{i_1} z_{i_2} \cdots z_{i_n}) = z_{i_1} z_{i_2} \cdots z_{i_j} z_m z_{i_{j+1}} \cdots z_{i_n}.$$

Since m is fixed, we do not write the dependence of r_j on m . Furthermore, we now allow $i_k = i_{k+1}$. Let $\beta \in \tilde{\ell}$. Then for any $\alpha \in \tilde{\ell}$ and for any $0 \leq j \leq |\alpha|$ such that $r_j(\alpha) = \beta$, we have $|\alpha| + 1 = |\beta|$ and

$$(2\mathbb{N})^\alpha = \prod_{l=1}^{|\alpha|} (2i_l^{(\alpha)}) = (2m)^{-1} \prod_{l=1}^{|\beta|} (2i_l^{(\beta)}) = (2m)^{-1} (2\mathbb{N})^\beta.$$

Moreover,

$$\begin{aligned} |\{(\alpha, j) : \alpha \in \tilde{\ell}, 0 \leq j \leq |\alpha|, r_j(\alpha) = \beta\}| &= \\ &= |\{1 \leq k \leq |\beta| : \beta = z_{i_1} \cdots z_{i_{|\beta|}}, i_k = m\}| \leq |\beta|. \end{aligned}$$

Thus, denoting $\tilde{\ell}_m = \{\beta \in \tilde{\ell} : \beta = z_{i_1} \cdots z_{i_{|\beta|}}, i_k = m \text{ for some } k\}$

$$\begin{aligned} \|D_m f\|_q^2 &= \sum_{\alpha \in \tilde{\ell}} \left| \sum_{j=0}^{|\alpha|} f_{r_j(\alpha)} \right|^2 (2\mathbb{N})^{-\alpha q} \\ &\leq \sum_{\alpha \in \tilde{\ell}} (|\alpha| + 1)^2 \sum_{j=0}^{|\alpha|} |f_{r_j(\alpha)}|^2 (2\mathbb{N})^{-\alpha q} \\ &= \sum_{\beta \in \tilde{\ell}_m} \sum_{\{(\alpha, j) : \alpha \in \tilde{\ell}, 0 \leq j \leq |\alpha|, r_j(\alpha) = \beta\}} (|\alpha| + 1)^2 |f_{r_j(\alpha)}|^2 (2\mathbb{N})^{-\alpha q} \\ &= \sum_{\beta \in \tilde{\ell}_m} \sum_{\{(\alpha, j) : \alpha \in \tilde{\ell}, 0 \leq j \leq |\alpha|, r_j(\alpha) = \beta\}} |\beta|^2 |f_\beta|^2 (2m)^q (2\mathbb{N})^{-\beta q} \\ &\leq \sum_{\beta \in \tilde{\ell}_m} |\beta|^3 |f_\beta|^2 (2m)^q (2\mathbb{N})^{-\beta q}. \end{aligned}$$

By induction it can be easily checked that for any $n \in \mathbb{N}$, $2^{3(n-1)} \geq n^3$.

Thus, for any $q \geq p + 3$ and for any $\beta \in \tilde{\ell}_m$,

$$(2m)^{-(q-p)} (2\mathbb{N})^{(q-p)\beta} = (2m)^{-(q-p)} (2i_1 \cdots 2i_{|\beta|})^{q-p} \geq 2^{3(|\beta|-1)} \geq |\beta|^3.$$

Therefore,

$$|\beta|^3 (2m)^q (2\mathbb{N})^{-\beta q} \leq (2m)^p (2\mathbb{N})^{-\beta p},$$

and we obtain

$$\|D_m f\|_q^2 \leq (2m)^p \|f\|_p.$$

In particular D_m is continuous.

It is now easy to check that for any $f, g \in \tilde{\mathcal{S}}_{-1}$ which are finite linear combinations of the basis (e_α) , $D_m(f \otimes g) = D_m(f) \otimes g + f \otimes D_m(g)$. By continuity it holds for any $f, g \in \tilde{\mathcal{S}}_{-1}$. \square

We recall that for a unital (associative) ring R a pair $(C, A) \in R^{p \times N} \times R^{N \times N}$ is called *observable* if there exists some $p \geq 0$ such that

$$(C \quad CA \quad CA^2 \quad \dots \quad CA^{q-1})$$

is left invertible. If furthermore, we may choose $q = N$, then we the pair (C, A) is called *strongly observable*.

In the following theorem and its proof we omit the symbol \otimes for simplicity.

Theorem 5.2. *Let $(C, A) \in \tilde{\mathcal{S}}_{-1}^{p \times N} \times \tilde{\mathcal{S}}_{-1}^{N \times N}$. If the pair $(E[C], E[A])$ is observable, then the pair (C, A) is observable.*

Proof. Let $q \geq 0$ be such that $(E[C] \quad E[C]E[A] \quad \dots \quad E[C]E[A^{q-1}])$ is left invertible. We show that for any $f \in (\tilde{\mathcal{S}}_{-1})^{qN}$ such that

$$(C \quad CA \quad \dots \quad CA^{q-1}) f = 0$$

it holds that $f = 0$.

First, we note that for such f , $(E[C] \quad E[C]E[A] \quad \dots \quad E[C]E[A^{q-1}]) E[f] = 0$. Hence, $f_0 = E[f] = 0$.

Now,

$$\begin{aligned} 0 &= (ED_m)((C \quad CA \quad \dots \quad CA^{q-1}) f) \\ &= (ED_m)(C \quad CA \quad \dots \quad CA^{q-1}) E[f] \\ &\quad + (E[C] \quad E[C]E[A] \quad \dots \quad E[C]E[A^{q-1}]) (ED_m)f \\ &= (E[C] \quad E[C]E[A] \quad \dots \quad E[C]E[A^{q-1}]) f_{z_m}. \end{aligned}$$

implies $f_{z_m} = 0$.

Furthermore, by a simple induction since there exist some $\{U_k\}_{k < n}$ such that

$$D_m^n((C \quad CA \quad \dots \quad CA^{q-1}) f) = \sum_{k < n} U_k D_m^n f + (C \quad CA \quad \dots \quad CA^{q-1}) D_m^n f$$

we conclude

$$\begin{aligned} 0 &= (ED_m^n)((C \quad CA \quad \dots \quad CA^{q-1}) f) \\ &= \sum_{k < n} E[U_k] E[D_m^n f] + (E[C] \quad E[C]E[A] \quad \dots \quad E[C]E[A^{q-1}]) (ED_m^n) f \\ &= (E[C] \quad E[C]E[A] \quad \dots \quad E[C]E[A^{q-1}]) f_{z_m^n}. \end{aligned}$$

Thus, $f_{z_m^n} = 0$, for any n and m .

The next step is to show that $f_{z_l z_m} = 0$. Since,

$$\begin{aligned}
0 &= (ED_l D_m)((C \ CA \ \cdots \ CA^{q-1}) f) \\
&= (ED_l)((C \ CA \ \cdots \ CA^{q-1}) E[D_m f] \\
&\quad + ((E[C] \ E[C]E[A] \ \cdots \ E[C]E[A^{q-1}]) E[D_l D_m f] \\
&\quad + (ED_l D_m)((C \ CA \ \cdots \ CA^{q-1}) E[f] \\
&\quad + (ED_m)((C \ CA \ \cdots \ CA^{q-1}) E[D_l f] \\
&= (E[C] \ E[C]E[A] \ \cdots \ E[C]E[A^{q-1}]) f_{z_l z_m}
\end{aligned}$$

we conclude that $f_{z_l z_m} = 0$.

In the same manner it is easy to complete the proof and showing that $f_\alpha = 0$ for any $\alpha \in \tilde{\ell}$. \square

In the approach outlined here to non-commutative linear systems we replaced the complex numbers by a non-commutative algebra with a special topological structure. Other approaches are possible. We mention in particular the work of Fliess [8]. We also mention [2, 6, 15, 13].

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